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Complex Bandpass Filter Banks for Signal Analysis

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ABSTRACT

In this paper a mathematical analysis based on the Complex Continuous Wavelet Transform over different theoretical signals is presented. The mathematical relationships between the Complex Continuous Wavelet Transform, the Hilbert Transform and complex filter banks are presented in order to obtain useful filter bank design parameters. The mathematical analysis of three different signals is presented: a pure cosine, a sum of cosines and a signal with frequency variations. The obtained theoretical results allow us to define a new algorithm to obtain an additive model of the input signal.

1. INTRODUCTION

The Complex Continuous Wavelet Transform was firstly introduced and applied to audio signals by Konland-Martinet et al. [1], [2], [3] and [4]. In recent years Carmona et al. [5] and [6] have developed new improvements. This work is mainly inspired in these ones.

The Complex Continuous Wavelet Transform (CCWT) of a signal $f(t)$ can be defined as follows:

$$W_f(s,k) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-k}{s}\right)dt \qquad (1)$$

Following Kronland-Martinet's et al. work [1] the mother wavelet we have used in this work is the complex Morlet's wavelet, which can be written as:

$$\psi(t) = Ce^{\frac{-t^2}{2\sigma^2}}\left(e^{j\omega_0 t} - e^{\frac{\omega_0^2}{2}}\right) \approx Ce^{\frac{-t^2}{2\sigma^2}}e^{j\omega_0 t} \qquad (2)$$

Here ω_0 and σ are control parameters, the central frequency and the filter width, respectively. Indeed, in the frequency domain, Morlet's wavelet is a typical bandpass filter. As shown in Mallat's work [7] this filter applied to equation 1 give us a family of filters (a filter bank) by its time-shifting and its dilation (or contraction) in frequency. Morlet's wavelet is presented in figure 1. For more information about Wavelet Analysis and Morlet's wavelet see [8].

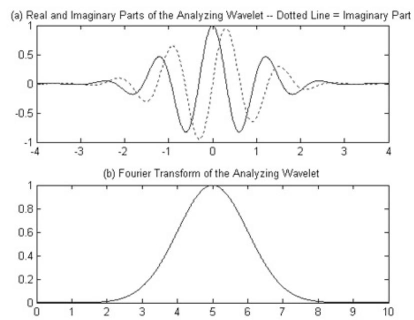

Fig. 1: Representation in time and frequency domain of the Morlet's Analyzing Wavelet, if $C = 1$, $\omega_0 = 5$ and $\sigma = 1$.

The whole information of $f(t)$ is inside the wavelet coefficients $W_f(s,k)$ obtained using equation 1. In this case, due to the complex nature of the mother wavelet they can be studied in terms of modulus and phase. But not all the information contained in the complex coefficients is necessary to appropriately analyze (and synthesize) the signal. As Kronland-Martinet et al and Carmona et al showed in [2], [5] and [6], the essential contribution to 1 is located around the points of the time-frequency half-plane where the modulus of such coefficients is maximum, the so called *ridge* of the transform. The ridge can be obtained by means of the *stationary phase argument* method, and the wavelet coefficients are only calculated in that set of points, obtaining the *skeleton* of the transform. The obtained results allow the reconstruction of the original signal with only a penalty function that can be reduced to a minimal noisy background. A different approach is made in this work. Here, the wavelet coefficients are firstly obtained, then the maxima points are located in the time-frequency half-plane and finally the phase of the coefficients on those points is calculated.

Let's see now a brief summary about the Hilbert Transform, the analytic signal and their relationships with the CCWT.

Let $f(t)$ be the signal to analyze. A general expression to refer such a signal is:

$$f(t) = A(t)\cos(\phi(t)) \tag{3}$$

For a given signal $f(t)$, expression 3 is far from being unique, but there is a pair of functions $A_f(t) \geq 0$ and $\phi_f(t) \in [0, 2\pi]$ called the *canonical pair* of $f(t)$. The canonical pair can be obtained computing the analytic signal related to $f(t)$ by:

$$Z_f(t) = f(t) + jF(t) \approx A(t)e^{j\phi(t)} \tag{4}$$

Where $F(t)$ is the HT of $f(t)$. This result is known as the Bedrosian's Theorem. Morlet's wavelet can be also seen as an analytic filter obtained from a real filter $g(t)$ defined as:

$$g(t) = Ce^{\frac{-t^2}{2\sigma^2}} cos(\omega_0 t) \tag{5}$$

Equation 1 can be seen as the temporal convolution between $f(t)$ and the analytic filter $\psi(t)$ or similarly as the convolution of the analytic signal and the real bandpass filter. Following the definition of the HT (see for example [9] and [10]) it can be obtained that:

$$W_f(s,k) = \begin{cases} 2g(\omega)f(\omega) & \text{if } \omega > 0 \\ g(0)f(0) & \text{if } \omega = 0 \\ 0 & \text{if } \omega < 0 \end{cases} \tag{6}$$

In equation 6 $f(\omega)$ and $g(\omega)$ are the Fourier Transform of $f(t)$ and $g(t)$, respectively. Equation 6 serve us to implement an FFT based efficient algorithm to compute the CCWT.

2. PRACTICAL LIMITS

If we want to properly analyze a signal, we must impose conditions to the time duration and bandwidth of the signal and the filter used to analyze it. These conditions, reflected in a relation between the control parameters ω_0 and σ, represent a practical limit for the analysis, and are mathematically and numerically defined in [11]. This limit can be expressed as:

$$BT > 5 \tag{7}$$

where B is a measure of the bandwidth and T a measure of time duration. Appliyng 7 to our filterbank it follows that the product $\sigma^2\omega_0^2$ has to be large enough, or numerically:

$$\sigma^2\omega_0^2 > 25 \tag{8}$$

This condition impose a practical limit to the definition of our filter bank control parameters and should be used properly in the implemented algorithm.

Another important aspect to be taken into account is the role played in equation 2 by the normalization constant C. It is usually chosen to normalize the filter energy, but in this work a normalization constant depending of the in signal is chosen. With this method the modulus of $W_f(s,k)$ can be directly the instant amplitude of $f(t)$ and its phase the signal's instant phase.

3. THE AM CASE

3.1. The simplest case: A pure cosine

Let's suppose that the signal is a simple cosine function with constant amplitude A_1 and angular frequency ω_1:

$$f(t) = A_1 \cos(\omega_1 t) \tag{9}$$

If we take the normalization constant as:

$$C = \sqrt{\frac{2}{\pi}}\frac{1}{s\sigma}, \tag{10}$$

and solving the Gaussian integral that appears in equation 1, the obtained wavelet coefficients are:

$$\begin{aligned} c(s,k) = 2A_1 e^{-\frac{\sigma^2\omega_1^2 s^2}{2}} e^{-\frac{\sigma^2\omega_0^2}{2}} \\ \big\{\cosh(\sigma^2\omega_0\omega_1 s)\cos(\omega_1 k) \\ +j\sinh(\sigma^2\omega_0\omega_1 s)\sin(\omega_1 k)\big\} \end{aligned} \tag{11}$$

It is possible to analyze equation 11 in terms of modulus and phase. The modulus of equation 11 can be expressed as:

$$\begin{aligned} \|c(s,k)\|^2 = 2A_1^2 e^{-\sigma^2\omega_1^2 s^2} e^{-\sigma^2\omega_0^2} \\ \big\{\cosh(2\sigma^2\omega_0\omega_1 s) + \cos(2\omega_1 k)\big\} \end{aligned} \tag{12}$$

It can be deduced that equation 12 has a maximum located at $s = \omega_0/\omega_1$ when $\sigma^2\omega_0^2$ is large enough, condition that carries out numerically if 8 is true. This is the same result offered by applying the stationary phase argument method developed in [2]. The maximum value of 12 is exactly A_1^2 due to the chosen value of C. Under these circumstances its achieved directly that the modulus of the coefficients is exactly the amplitude of the original signal, with no dependence of the scale factor. A graphical view of equation 12 is shown in figure 2.

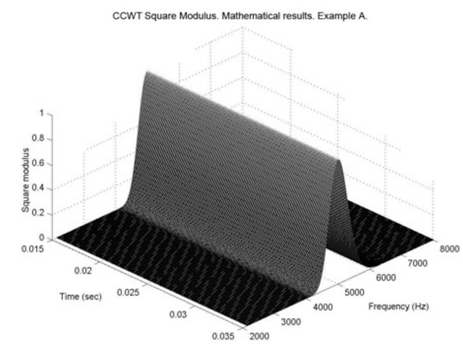

Fig. 2: Square modulus of the wavelet coefficients for a pure cosine. $A_1 = 1$ and $\omega_1 = 5$ kHz. Observe that the maximum is exactly located at the appropriate frequency and its value is 1.

The phase of equation 11 can be expressed as:

$$\Phi(s,k) = \arctan[\tanh(\sigma^2\omega_0\omega_1 s)\tan(\omega_1 k)] \tag{13}$$

If $s = \omega_0/\omega_1$ and equation 8 is true, the temporal partial derivative of equation 13, that is the instantaneous angular frequency, is:

$$\frac{\partial[\Phi(s,k)]}{\partial k} = \omega_1 \tag{14}$$

So, the amplitude and the phase of the original signal are perfectly recovered.

3.2. The sum of n pure cosines

Let's suppose that the analyzing signal is made up of a sum of n cosines, each one with constant amplitude A_α and pure frequency ω_α.

$$f(t) = \sum_{\alpha=1}^{n} A_\alpha \cos(\omega_\alpha t) \tag{15}$$

Proceeding as in the previous case, this time its necessary to calculate n Gaussian integrals. The complex wavelet coefficients are:

$$\begin{aligned} c(s,k) = & \sum_{\alpha=1}^{n} 2A_\alpha e^{-\frac{\sigma^2\omega_\alpha^2 s^2}{2}} e^{\frac{\sigma^2\omega_0^2}{2}} \\ & \big\{\cosh(\sigma^2\omega_0\omega_\alpha s)\cos(\omega_\alpha k) \\ & +j\sinh(\sigma^2\omega_0\omega_\alpha s)\sin(\omega_\alpha k)\big\} \end{aligned} \tag{16}$$

As made before, it can be obtained the modulus and phase of the wavelet coefficients.

$$\begin{aligned} & \|c(s,k)\|^2 = 2e^{-\sigma^2\omega_0^2}\Bigg\{\sum_{\alpha=1}^{n} 2A_\alpha^2 e^{-\sigma^2\omega_\alpha^2 s^2} \\ & \big[\cosh(2\sigma^2\omega_0\omega_\alpha s)+\cos(2\omega_\alpha k)\big]+ \\ & \sum_{\alpha\neq\beta=1}^{n} A_\alpha A_\beta e^{-\frac{\sigma^2(\omega_\alpha^2+\omega_\beta^2)s^2}{2}} \\ & \Bigg[\cosh[\sigma^2\omega_0(\omega_\alpha+\omega_\beta)s]\cos[(\omega_\alpha-\omega_\beta)k]+ \\ & \cosh[\sigma^2\omega_0(\omega_\alpha-\omega_\beta)]\cos[(\omega_\alpha+\omega_\beta)k]\Bigg]\Bigg\} \end{aligned} \tag{17}$$

$$\begin{aligned} & \Phi(s,k) = \\ & \arctan\frac{\sum_{\alpha=1}^{n} A_\alpha e^{-\frac{\sigma^2\omega_\alpha^2 s^2}{2}}\sinh(\sigma^2\omega_0\omega_\alpha s)\sin(\omega_\alpha k)}{\sum_{\alpha=1}^{n} A_\alpha e^{-\frac{\sigma^2\omega_\alpha^2 s^2}{2}}\cosh(\sigma^2\omega_0\omega_\alpha s)\cos(\omega_\alpha k)} \end{aligned} \tag{18}$$

The modulus and phase behavior is similar as it was shown in the upper paragraphs. The exact shape of equation 17 is shown in figures 3 and 4, with two different original signals composed by two and three cosine waves, respectively.

Expressions given in equations 17 and 18 require a more detailed explanation. It can be shown that 17 have maxima located exactly over every scale factor $s = s_\alpha = \omega_0/\omega_\alpha$. In these points, square modulus is exactly A_α^2 and instant angular frequency of 18 is exactly ω_α. But if the frequencies involved are closer

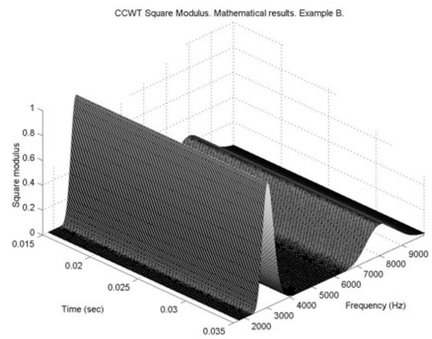

Fig. 3: Two cosines, with $A_1 = 1$, $\omega_1 = 3$ kHz, $A_2 = 0.5$ and $\omega_2 = 8$ kHz. Observe the maxima locations at the right frequency and their correct value.

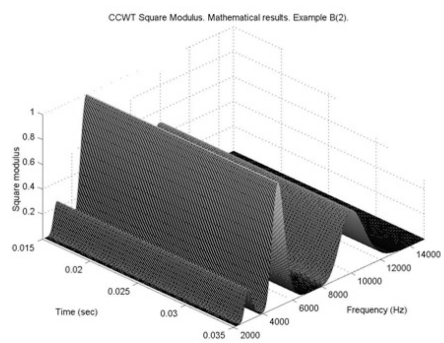

Fig. 4: Three cosines, with $A_1 = 0.5$, $\omega_1 = 3$ kHz, $A_2 = 1$ and $\omega_2 = 5$ kHz, $A_3 = \sqrt{0.5}$, and $\omega_3 = 10$ kHz.

enough, there start to increase some intermodulation terms that alter the shape of the square modulus, as can be seen in figure 5.

The mathematical analysis made up to now give us some conclusions. If the analyzing signal is made up of a sum of cosines and locating the maxima of the modulus of the wavelet complex coefficients with the proper normalization constant we can obtain the correct amplitude and phase values of the original signal. When the sinusoids are close enough some intermodulation terms appear, and to resolve properly them the filter bank width should be narrowed.

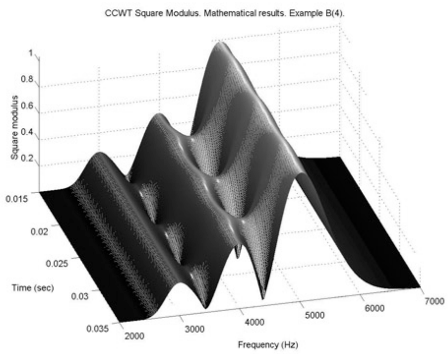

Fig. 5: Three cosines, with $A_1 = 0.5$, $\omega_1 = 3$ kHz, $A_2 = \sqrt{0.5}$ and $\omega_2 = 8$ kHz, $A_3 = 1$, and $\omega_3 = 10$ kHz. Observe in this case the intermodulation terms.

4. THE FM CASE

Let's suppose now that $f(t)$ has a more or less complicated modulation frequency law. Due to the possibility of performing a mathematical analytic analysis a quadratic law has been chosen. Let the analyzing signal be:

$$f(t) = A_1 \cos(at^2 + bt + c) \tag{19}$$

Calculating again the Gaussian integral of equation 1 we can obtain the wavelet coefficients and, then, their modulus and phase:

$$\begin{aligned}\|c(s,k)\|^2 =& \frac{C^2 A_1^2}{2} \frac{2\pi\sigma^2 s^2}{\sqrt{1+4a^2\sigma^4 s^4}} e^{-2\xi(s,k)} \\ & \{\cosh[2u(s,k)] + \cos[2\nu(s,k)]\}\end{aligned} \tag{20}$$

$$\Phi(s,k) = \arctan\{\tanh[u(s,k)]\tan[\nu(s,k)]\}, \tag{21}$$

where functions ξ, u and ν can be expressed as:

$$\begin{aligned}&\xi(s,k) = \\ &\frac{4k^2a^2\sigma^2s^2 + (\omega_0 + s^2b^2)\sigma^2 + 4kabs^2\sigma^2}{1+4a^2\sigma^4s^4}\end{aligned} \tag{22}$$

$$u(s,k) = \frac{\sigma^2\omega_0 s(2ak+b)}{1+4a^2\sigma^4s^4} \tag{23}$$

$$\begin{aligned}&\nu(s,k) = \\ &\frac{c(1+4a^2\sigma^4s^4) + ak^2 - (\omega_0^2 + s^2b^2)\sigma^4s^2a + kb}{1+4a^2\sigma^4s^4} + \\ &\frac{arctan(2a\sigma^2s^2)}{2}\end{aligned} \tag{24}$$

Taking $a = 0$ and $c = 0$ it can be seen that the examples studied in the previous section are only particular cases of this one. If the normalization constant is:

$$C = \sqrt{\frac{2(1+4a^2\sigma^4s^4)^{1/2}}{\pi\sigma^2s^2}} \tag{25}$$

when the scale is:

$$s = s_k = \frac{\omega_0}{2ak+b}, \tag{26}$$

it is obtained that the maximum of the modulus of the wavelet coefficients is:

$$\|c(s,k)\|^2 = A_1^2, \tag{27}$$

and the phase

$$\phi(s,k) = ak^2 + bk + c \tag{28}$$

The evolution across scales of the modulus of the wavelet coefficients is depicted in figure 6.

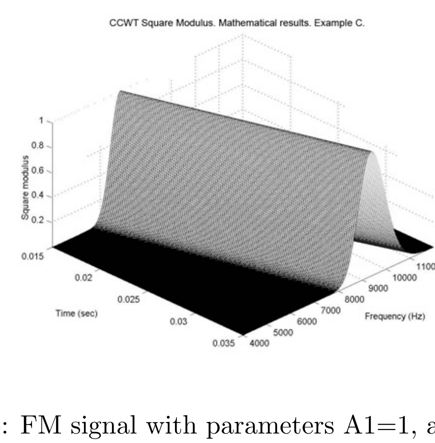

Fig. 6: FM signal with parameters A1=1, a=62500, b=5000, c=20. Observe the frequency modulation law.

This analysis make clear that its not enough to locate the maximum of the module at some constant scale but its necessary to follow the location of the maximum in time and frequency. In the case of a quadratic frequency dependence the path is defined by equation 26. This is the same result that the one obtained in Kronland-Martinet's et al. work using the stationary phase argument.

5. CONCLUSIONS

In this work the main basis for the development of a wavelet based signal analysis algorithm are settled. The computing algorithm has been clarified in terms of an FFT implementation looking at the relationships between the Hilbert Transform and the Complex Continuous Wavelet Transform. The relationships between the control parameters of the filter bank are also expressed.

Analyzing simple cosine signals we have defined the normalization constant needed to obtain physical meaning results. In these simple signals we obtain the correct amplitudes and phases of the original signals looking at the points in which the modulus of the transform coefficients have a maximum. We have obtained a correct amplitude and phase representation of a frequency dependent signal following the maxima of the modulus of the transform in the time-frequency half-plane. The analysis of this frequency dependent signal shows us that the developed algorithm should include, a priori, some signal-dependent normalization constant.

The results obtained in this work will serve us to develop a signal analysis algorithm that will describe an audio signal in terms of its time-dependent amplitude and phase constituent partials, like in a general additive synthesis model.

6. ACKNOWLEDGEMENTS

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7. REFERENCES

[1] R. Kronland-Martinet, J. Morlet, and A. Grossmann, "Analysis of sound patterns through wavelet transforms," *Int. J. of Patt. Recog. and Artif. Intell*, vol. 1, no. 2, pp. 272–302, 1987.

[2] R. Kronland-Martinet, "The Wavelet Transform for Analysis, Synthesis and Processing of Specch and Music Sounds," *Computer Music Journal*, vol. 12, no. 4, 1988.

[3] P. Guillemain and R. Kronland-Martinet, "Characterization of Acoustic Signals through Continous Linear Time-Frequency Representations," *Proceedings of the IEEE*, vol. 84, no. 4, Apr. 1996.

[4] N. Delprat, B. Escudié, P. Guillemain, R. Kronland-Martinet, P. Tchamitchian, and B. Torrésani, "Asymthotic Wavelet and Gabor Analysis: Extraction of Instantaneous Frequencies," *IEEE Tran. on Inform. Theory*, vol. 38, no. 2, Mar. 1992.

[5] R. A. Carmona, W. L. Hwang, and B. Torrésan, "Characterization of signals by the ridges of their wavelet transforms," *IEEE Transactions on Signal Processing*, vol. 45, no. 10, pp. 2586–2590, Oct. 1997.

[6] R. A. Carmona, W. L. Hwang, and B. Torrésan, "Multiridge detection and time-frequency reconstruction," *IEEE Transactions on Signal Processing*, vol. 47, no. 2, pp. 480–492, Feb. 1999.

[7] S. Mallat, "A Theory for Multiresolution Signal Decomposition: The Wavelet Representation," *IEEE Tran. on Patt. Anal. and Machine Intell.*, vol. 11, no. 7, pp. 674–693, 1989.

[8] I. Daubechies, *Ten Lectures on wavelets*, vol. 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*, SIAM, 1992.

[9] S. L. Hahn, "Short Communicaction on the Uniqueness of the Definition of the Amplitude and Phase of the Analytic Signal," *Signal Processing*, vol. 83, pp. 1815–1820, 2003.

[10] D. G. Long, *Comments on Hilbert Transform Based Signal Analysis*, Brigham Young University, 1998.

[11] B. Boashash, "Estimating and Interpreting the Instantaneous Frequency of a Signal. Part 1: Fundamentals," *Proceedings of the IEEE*, vol. 80, no. 4, Apr. 1992.